On the Convergence Speed in Growth Models*

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J. Philipp Reiss
Otto von Guericke University of Magdeburg
Faculty of Economics & Management
PSF 4120
39016 Magdeburg/Germany
reiss@ww.uni-magdeburg.de

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Abstract

The growth literature is concerned with the convergence speed of economies providing valuable information on the speed at which out-of-steady state-economies move towards their long-run equilibria. Prominent examples of papers on that topic include Mankiw/Romer/Weil (1992) and Barro/Sala-i-Martin (1992) both empirically oriented, and Jones (1995) and Ortigueira/Santos (1997) being theoretically focused. A common shortcoming of such studies is that any convergence speed measure employed is usually derived by linearization around the steady state. However, knowledge of the convergence speed is important only if an economy is not already in the vicinity of its long-run equilibrium. The current paper presents a method, which allows to quantify the occuring error due to linearization in any growth model featuring stable steady states. Neoclassical growth theory is used to introduce the methodology itself. Applications to some growth models are presented, too.

Keywords: Convergence Speed, Growth Theory, Transitional Dynamics, Linearization Error

***JEL classification*:** O40, O41, C69

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1 Introduction

R. Sato (1963) was the first to inquire into the time period required for an economy to overcome a certain amount of a displacement from its steady state. In particular, he investigated fiscal policy in the Solow model: R. Sato took the fixed investment ratio as a policy variable and analyzed the adjustment speed[1] of a policy change. For a set of parameter values[2], he found that the exact time period for a 90 per cent adjustment is quite long - taking 100 years. Since complete adjustment is never achieved within finite time, R. Sato uses the arbitrary value 90% of a variable's steady state level[3] to characterize its *vicinity* to the new long-run equilibrium. Three years later K. Sato (1966) challenged this result. He noted that the long adjustment periods calculated by R. Sato were due to his assumptions of a zero-depreciation rate of capital goods[4] and pure disembodied technical progress. Using K. Sato's suggested depriciation rate of 8% (without introducing technical advance of the embodied type), for R. Sato's parameter set yields an adjustment time of only 30 years. With the diminishing interest of the economic profession in growth economics, the convergence speed debate then came to a halt. It was revived by the empirical papers Mankiw/Romer/Weil (1992) and Barro/Sala-i-Martin (1992) which both estimate the approximated convergence speed. It was found to be quite small, standing in stark contrast to calibrated predictions of the underlying growth models. Soon after, the applied econometric methodology was questioned on several grounds and subsequent papers report higher estimates of convergence speeds. While that renewed debate is rather empirical and solely concerned with a measure of the adjustment speed derived in the vicinity of the steady state, Ortigueira and Santos (1997) discuss steady state convergence in neoclassical and endogenous growth models[5]. They question the validity of empirical measures given *off-steady-state*. However, they only provide a graphical solution.

[1] The terms *convergence*, *adjustment*, and *transition speed* are used interchangeably.

[2] Sato (1963) worked with a convential Cobb-Douglas production function and the values: initial/new investment rate: 11.66%/12.54%; capital share: 0.35; labor force growth rate: 1.5%; rate of labor-augmenting technical progress: 2%.

[3] Notice that the steady state level remains constant over time. The variable for comparison - stability criterion in his terms - was the average product of capital.

[4] Or equivalently due to an investment policy of the form $I = s(Y - \delta K)$, where Y is *gross* output. Worn out capital, δK, is immediately replaced by new capital goods, and additionally, a constant fraction s of *net* output is invested, see K. Sato (1966, p. 265).

[5] Specificially, the standard Ramsey model makes up the neoclassical part while human capital models á la Uzawa (1965) and Lucas (1988) stand in for new growth theory.

This demonstrates that the laws of motions of per-capita capital obtained by numerical solution of the nonlinear systems and linearization around the long-run equilibrium seem to coincide over a large range of the per-capita capital stock.

As a common characteristic the speed of transition of off-steady-state-economies is measured using linearization around the long-run equilibrium. Strictly speaking, those measures are valid only if the model economy is in steady-state-neighborhood. Yet, the speed of transition only bears importance, if the economy is not in steady state vicinity. Nevertheless, exemplified by Jones (1995, p. 774), it seems to be the standard procedure by now to start off with linearization in order to obtain a constant coefficient measuring the speed of convergence.

This paper takes up that problem and develops a method which allows to assess whether the errors introduced by linearization are serious or rather negligible. It complements Ortigueira/Santos (1997) in providing quantifications of the linearization error's size instead of arguing on the basis of graphical impressions.

There are at least five reasons why the convergence speed matters. First, if the adjustment speed is high, the economy is almost always close to its long-run equilibrium. Therefore, it is sufficient to be solely concerned with steady state behavior and comparative dynamics of the growth model. On the other hand, if the speed of transition is low, it is situated in some distance to its steady state most of the time. Transitional dynamics then become very important because they exclusively provide a reasonable description of the model's growth process. Hence, the convergence coefficient may serve as an indicator whether steady state behavior or transitional dynamics should be emphasized by a model's analysis. If a growth theory is required to exhibit steady state growth, a high convergence speed appears to constitute a necessary condition for it. Moreover, the convergence coefficient provides an additional empirically testable hypothesis. If the adjustment speed is estimated, a growth model's convergence coefficient's calibration can be compared to its estimate. This will provide another source of evidence concerning the particular growth model at hand[6]. Further, once reliable estimates are available, a growth model's calibra-

[6] By now, there exists a multitude of papers which deal with the estimation of the adjustment speed. They differ in their applied methodology (cross-country regression vs. panel data approach) and in their results. Most cross-country regression papers, e.g. Barro/Sala-i-Martin (1992), report an estimate of 2% which seems to be robust over various data sets. However, another part of that literature challenges this result by questioning the applied method of estimation. For a critical discussion compare Ortigueira/Santos (1997, p. 383f.)

tion exhibits an additional degree of freedom. Using the additional information provided by the coefficient's estimate, parameters may be inferred from the model's structure, the coefficient's estimate, and proxies of more robust parameters[7]. Finally, if a growth model is thought to capture the growth process of a real economy, its adjustment speed may be determined and calibrated, providing valuable information on the time that economy needs to bridge any eventual gap to its steady state position. For example suppose Germany could be portrayed by such a model and a constant convergence speed of 7% be derived. If the western part is in equilibrium while the eastern part is not, it could be concluded that the eastern part needed around ten years to close half of the gap.

The remainder of this paper is organized as follows: Section 2 introduces the convergence coefficient and discusses some of its general properties. The quality-quantification-method is introduced in section 3. In section 4 it is applied to the Solow model. It is demontrated that the elasticity of substitution between capital and labor is crucial for the quality of the linearization. Section 5 contains an application to the Ramsey model. Final conclusions are provided in section 6.

2 The Convergence Coefficient

In this section we define the convergence coefficient and discuss two properties which are widely neglected. Let $Y(t)$ denote the differentiable time-path of any variable converging to its balanced growth equilibrium, Y^*, which is constant over time.

Definition 1 *Y's speed of convergence is given by* $\beta_Y(t) \equiv -\frac{d(Y^*-Y(t))/dt}{Y^*-Y(t)}$.

Thus, β_Y (roughly) measures by how many percentage points Y's steady state gap, Y^*-Y, vanishes as an infinitesimal unit of time elapses[8]. This is the traditional definition of the convergence coefficient as presented in e.g. K. Sato (1966, p. 264) and Gandolfo

[7]Barro/Sala-i-Martin (1995, p. 37f.) apply this reasoning to the capital share utilizing the Solow model: they note that the (approximated) convergence speed may be given by $(1-\alpha)(n+\delta+x)$ – (α - capital share, n - labor force growth, δ - depreciation rate, x - labor-augmenting technical progress) – and that its estimate is 2%. Moreover, they fix { $n=0.01$, $\delta=0.05$, $x=0.02$ } and infer that the capital share must be $\alpha=0.75$ which is too large for a narrow concept of capital confined to physical capital. On the basis of this argument, they suggest that human capital plays an important role and that capital's conventional view should be broadened.

[8]To be precise, β_Y is the rate at which the steady state displacement declines relative to its size as time goes by.

(1996, p. 181ff.) Note that it deviates from the one contained in Barro/Sala-i-Martin (1995, p. 53) who define it to be a semi-elasticity of Y's growth rate with respect to its level, specificially $\beta_Y^{BS} \equiv -\frac{d(\dot{Y}/Y)}{d\log Y}$. It measures by how much Y's growth rate decreases as its level is increased by one per cent. Since β^{BS} is neither related to convergence behavior nor exhibits any time dimension, this paper appeals to definition 1.

Due to the fact that Y^* is a constant, an equivalent way of defining the speed of transition is:

Definition 2 *(Alternative Definition) Y's speed of convergence is equivalently given by* $\beta_Y(t) \equiv \frac{\dot{Y}(t)}{Y^*-Y(t)}$ *where* $\dot{Y} \equiv dY/dt$.

Departing from definition 2, the convergence speed's behavior of any variable Y with a constant steady state may be easily determined graphically if its equation of motion is an ordinary differential equation of first order, explicitly given, and its rate of change doesn't depend on other variables but on Y. As an example consider the capital accumulation equation of a Solow model given in standard notation by

$$\dot{k} = sf(k) - (n+\delta)k.$$

Figure 1a depicts $\dot{k}$'s shape if the underlying production function conforms to the Inada-conditions. For any capital stock per worker, k_o, the adjustment speed is given by the (negated) slope of the line arising if the point $(k_o, \dot{k}_o)$ is connected with the steady state, i.e. $\beta_k = \frac{\dot{k}}{k^*-k} = \tan\alpha$.

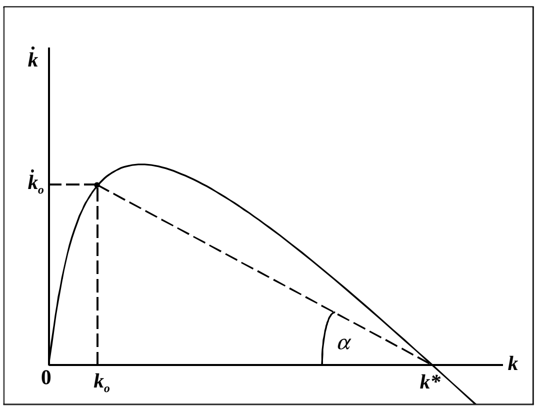

Fig. 1a: $\beta_k = \tan\alpha$

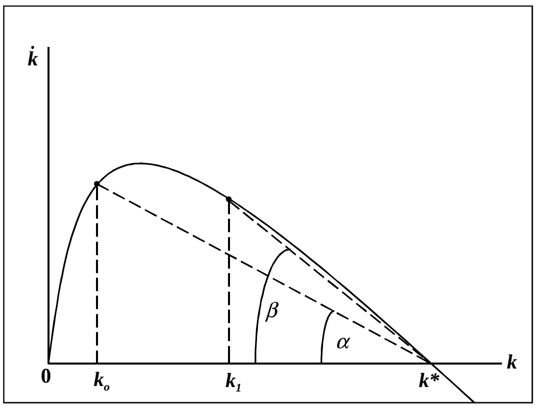

Fig. 1b: $k_o < k_1$, $\alpha < \beta$

Figure 1b illustrates that if the capital stock is below its steady state value, the convergence speed of k increases as the capital stock per worker moves closer to its steady state, $k_1 > k_o$, since the absolute value of the associated slope increases. In the limiting case when $k \to k^*$, application of l'Hôpital's rule to β_k shows that the convergence speed is given by (the negative of) $\dot{k}$'s derivative with respect to k evaluated at k^*, i.e. $\beta_k(k^*) = -\frac{d\dot{k}}{dk}(k^*)$. Obviously the convergence speed along a transition path is not necessarily constant which actually seems to be the rare exception. This is summarized in the following proposition.

Proposition 3 *In general any variable's convergence speed off-steady state is not constant over time. It usually changes along a transition path towards its long-run equilibrium.*

Figure 1 hints at the fact that $\dot{Y}$'s curvature determines the qualitative behavior of Y's convergence speed. The next proposition makes this link explicit.

Proposition 4 *Let $X(t)$ be differentiable, strictly monotonic and converging to its finite long-run equlibrium X^* defined by $\dot{X}|_{X=X^*} = 0$.*
(a) For any $X(0) < X^*$, the convergence speed of X, β_X, $\left\{ \begin{array}{c} \text{decreases} \\ \text{increases} \end{array} \right\}$ over time if $\dot{X}$ is strictly $\left\{ \begin{array}{c} \text{convex} \\ \text{concave} \end{array} \right\}$ in X. (b) For $X(0) > X^*$, the qualitative behavior of β_X over time as described in (a) is reversed. (c) If $\dot{X}$ is linear in X then β_X remains constant over time.

Proof (a): From the strict convexity of $\dot{X}$ it follows that $\lambda\dot{X}(X^*) + (1-\lambda)\dot{X}(X_o) > \dot{X}(\lambda X^* + (1-\lambda)X_o)$ for any $X_o \equiv X(t_o)$ and $\lambda \in (0,1)$. Since $\dot{X}(X^*) = 0$ and by the definition $X_1 \equiv X(t_1) \equiv \lambda X^* + (1-\lambda)X_o$ this relation can be rearranged such that $\dot{X}(X_o) > \frac{1}{1-\lambda}\dot{X}(X_1)$. Division by $X^* - X_o$ and making use of X_1 leads to $\frac{\dot{X}(X_o)}{X^*-X_o} \gtrless \frac{\dot{X}(X_1)}{X^*-X_1}$ as $X^* \gtrless X_o$. By the definition of the convergence coefficient and identities X_o/X_1 it follows that $\beta_X(t_o) \gtrless \beta_X(t_1)$ as $X^* \gtrless X_o$. Since $|X^* - X_o| > |X^* - X_1|$ it follows from the monotonic convergence behavior of $X(t)$ that $t_o < t_1$. This proves part (a) of the claim. (b), (c): The arguments are analogous to the preceding and therefore omitted. $\square$

The Solow model can also be used to highlight that different variables, such as output or capital, may converge at different transition speeds toward their long-run equilibria. If the production function is Cobb-Douglas, the equation of motion of output per worker, y, may be written as

$$\dot{y} = \alpha \left[s y^{\frac{2\alpha-1}{\alpha}} - (n+\delta)y \right]. \tag{1}$$

If the capital share α is smaller than 0.5, $\dot{y}$ is strictly convex in y and $y(0) < y^*$, it follows that output's convergence speed gradually falls as the economy adjusts towards its steady state. On the contrary, capital's adjustment speed increases along its transition path as noted above. Therefore we can state the next proposition:

Proposition 5 *Within the same growth model with the same parameter configuration, different variables, such as capital, output, or capital's average product, generally exhibit different convergence speed behavior.*

3 A Method to Quantify the Quality of Convergence Speed Approximations

Due to complicated non-linear dynamics, a measure of the convergence speed is usually derived by the system's linearization around its long-run equilibrium. This may imply an approximation error if the model economy is not in steady state-vicinity. In this section I present a method which allows to quantify the quality of such approximations.

3.1 The Approximation Error Due to Linearization

Let $\tilde{\beta}$[9] denote the approximation of the true convergence coefficient obtained by linearization. For ease of the exposition, the Solow model is used to illustrate the difference between the true value of β and its approximation, $\tilde{\beta}$. Specificially consider per capita output's equation given by (1). In order to derive an approximation to β_y, linearize equation (1) at its steady state, $y^* = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$, to obtain

$$\dot{y} \cong -(1-\alpha)(n+\delta)(y-y^*). \tag{2}$$

By the convergence coefficient's definition 2 and (2):

$$\tilde{\beta}_y = (1-\alpha)(n+\delta).$$

Hence, the approximated convergence coefficient is a constant. In contrast, its true counterpart, β_y, is a function of the economy's relative income position $\lambda \equiv y/y^*$. To see this, divide (1) by $y^* - y$. Upon some algebraic transformations it follows that

$$\beta_y = \alpha(n+\delta)\frac{\left(\frac{1}{\lambda}\right)^{\frac{1-\alpha}{\alpha}} - 1}{\frac{1}{\lambda} - 1}.$$

[9] The '$\tilde{}$' above β indicates that $\tilde{\beta}$'s derivation involved a first-order Taylor expansion.

Note that β_y is a function of all parameters of the model since λ depends negatively on y^* implying that $\lambda = \lambda(\underset{(-)}{s}, \underset{(-)}{\alpha}, \underset{(+)}{n}, \underset{(+)}{\delta})$. Only in the limiting case, $\beta_y \to \tilde{\beta}_y$ as $y \to y^*$. Figure 2 depicts output's true and approximated convergence speed for a capital share smaller than 0.5. Obviously, there exists an error due to linearization. If the economy's income is lower than its steady state-value then the approximation underestimates the transition speed. If actual income exceeds its long-run equilibrium then the economy converges slower to its steady state than the approximation predicts.

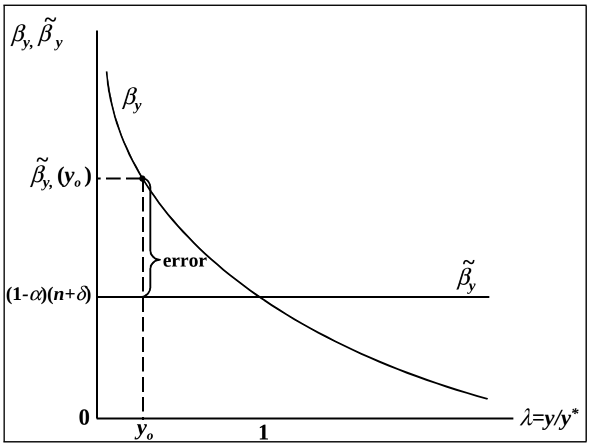

Fig. 2

3.2 Quantification of Linearization Error

It would be straightforward to calculate the difference between β and $\tilde{\beta}$ in order to quantify the approximation error. Yet, this figure would only characterize the linearization error at a single point on the adjustment path. Hence, it is questionable whether this error's observation is representative. Furthermore, gauging the severity of the linearization error may be a problem since the approximation's quality depends on the error's magnitude relative to the true convergence speed. Due to these difficulties, the following uses convergence coefficients to calculate *adjustment times* and compares approximated adjustment times basing on linearization, $\tilde{T}$, to the appropriate true ones, T, in order to obtain a measure of relative loss μ.

In a balanced growth equilibrium, all variables of a model grow at constant rates by definition. If the associated model economy happens to follow an off-equilibrium path which eventually converges to its balanced growth path, the model's variables will not grow at constant rates, but rather retain that property once the equilibrium path is reached. It would be natural to call that period of time needed for those variables to return to their steady state paths *adjustment time*. Unfortunately this is meaningless since complete restoration of steady states is in many growth models never achieved within finite time. To circumvent this problem, the literature uses the term adjustment time as length of that time period it takes a variable to reach a particular percentage level, say 95%, of its steady state-value. Equivalently *adjustment time* may be defined as that period of time it takes an economy to close a defined fraction of a variable's steady state gap. From the latter definition, a prominent special case follows. The gap's half-life may be interpreted as adjustment time. A variable X's half-life is that period of time needed until X's steady state gap is halved. In the following, half-lives of steady state gaps define T and $\tilde{T}$. This is the measure the literature relies on in order to illustrate the implications of specific transition speeds[10]. The steady state gap's half-life T_X of any variable $X(t)$ converging from its starting position $X_o = X(0) \neq X^*$ to its constant steady state is determined by the condition

$$X^* - X(T_X) = \frac{X^* - X_o}{2}. \tag{3}$$

If $X(t)$ is strictly monotonic (assumed henceforth), X may be inverted such that T_X is given by

$$T_X = X^{-1}\left(\frac{X^* + X_o}{2}\right).$$

Similarly the approximated half-life is

$$\tilde{T}_X = \tilde{X}^{-1}\left(\frac{X^* + X_o}{2}\right),$$

where $\tilde{X}(t)$ is X's approximated time path obtained by linearization around X^*. If the approximated convergence speed happens to be constant it is straightforward to show that the approximated half-life is given by[11]

$$\tilde{T}_X = \frac{\ln 2}{\tilde{\beta}} \quad \text{if } \tilde{\beta}_X = \text{const.} \tag{4}$$

As a measure of relative loss which is employed as the quality indicator of the approximation, I define the half-life error as

$$\mu \equiv \frac{\tilde{T}_X - T_X}{T_X}.$$

The quality of the approximation increases as the absolute value of μ falls. If $\mu = 0$, no approximation error occurs. Positive values of μ indicate by how many per cent the

[10] Cf. Jones (1995, p. 775) or Williams/Crouch (1972, p. 555).

[11] To see this note that in this case $X(t) = (X_o - X^*)e^{-\beta t} + X^*$ where $X(0) = X_o$. Substitution of this expression into half-life condition (3) and solving for $\tilde{T}_X$ proves the claim.

approximated half-life prediction exceeds the true time needed for adjustment, negative values of μ point to an underestimate of the true half-life. E.g. if $\mu = 1$ the half-life prediction based on the approximation overestimates the adjustment time by 100% of the true half-life.

3.3 Implementation Issues

A crucial element in the quantification process is the knowledge of true adjustment times. Since the reason for linear approximation is analytical intractability, numerical methods have to be invoked to find exact half-lives. Discussion of this matter is beyond the scope of this paper[12]. The following sections contain additional information if appropriate. Moreover, software code facilitating the analysis of those models using the computer package Maple are available on my homepage[13].

4 Application to a Solow Model

In the next subsections I quantify the approximation error in a Solow model. At first a Cobb-Douglas production function is adopted, afterwards it is replaced by a CES-production function. It is shown that for reasonable parameter values the approximated convergence speed is only of good quality if the elasticity of substitution is not much smaller than unity.

As pointed out in section 2, different variables have different (exact) convergence speeds and therefore differ also in their true half-lifes although the approximated convergence speed may be the same. Therefore a variable has to be chosen which serves as a stability criterion. In the following, the analysis focuses on per capita income y.

4.1 Cobb-Douglas-Production Function

In a Solow model with a Cobb-Douglas production function, all time paths can be solved for analytically because the fundamental equation of capital accumulation is a Bernoulli equation. The same is true if it is rewritten in terms of per capita income as inspection of (1) reveals which is repeated here for the reader's convenience

$$\dot{y} = \alpha \left[s y^{\frac{2\alpha-1}{\alpha}} - (n+\delta) y \right]. \tag{5}$$

[12] A good introduction into numerical analysis of economic models is provided by Judd (1998).

[13] http://www.ww.uni-magdeburg.de/vwl2/reiss.html

If $y(t)$ is combined with the half-life condition (3), the exact half-life of per capita income is given by[14]

$$T_y = \frac{\ln\left[d\left(1-\lambda_o^{\frac{1-\alpha}{\alpha}}\right)\right] - \ln\left\{d\left[1-\left(\frac{1+\lambda_o}{2}\right)^{\frac{1-\alpha}{\alpha}}\right]\right\}}{(1-\alpha)(n+\delta)}, \qquad d = \begin{cases} 1 & \text{f. } \lambda_o \in (0,1) \\ -1 & \text{f. } \lambda_o > 1 \end{cases}. \tag{6a}$$

Since the convergence speed $\tilde{\beta}_y = (1-\alpha)(n+\delta)$ is constant, by (4) the approximated half-life is

$$\tilde{T}_y = \frac{\ln 2}{(1-\alpha)(n+\delta)}. \tag{7}$$

It is easy to verify that the relative loss μ is given by

$$\mu = \frac{\ln 2}{\ln\left[d\left(1-\lambda_o^{\frac{1-\alpha}{\alpha}}\right)\right] - \ln\left\{d\left[1-\left(\frac{1+\lambda_o}{2}\right)^{\frac{1-\alpha}{\alpha}}\right]\right\}} - 1, \qquad d = \begin{cases} 1 & \text{f. } \lambda_o \in (0,1) \\ -1 & \text{f. } \lambda_o > 1 \end{cases} \tag{8}$$

The loss due to linearization obviously depends on two parameters: the capital share α and output's initial position relative to its steady state, λ_o. However, except for the capital share, the half-life error does not depend on the specific parameter configuration used.[15] Table 1 shows half-life errors for some combinations of α and λ_o.

[14] See the appendix for the full derivation.

[15] If labor-augmenting technical progress is introduced and/or the production function features a scale parameter ($y = Ak^{\alpha}$), this result remains unchanged.

$\lambda_o \setminus \alpha$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.10	14916.2	623.0	147.4	40.9	0	-20.3	-32.1	-39.6	-44.8
0.20	6743.7	405.2	104.9	30.5	0	-15.9	-25.5	-31.8	-36.3
0.30	3215.0	267.7	76.1	23.1	0	-12.6	-20.5	-25.8	-29.7
0.40	1593.6	178.8	55.6	17.6	0	-10.0	-16.4	-20.8	-24.1
0.50	810.9	119.5	40.2	13.2	0	-7.8	-12.9	-16.5	-19.1
0.60	417.3	78.58	28.4	9.7	0	-5.9	-9.8	-12.6	-14.7
0.70	211.8	49.5	19.1	6.7	0	-4.2	-7.0	-9.0	-10.6
0.80	100.4	28.3	11.5	4.1	0	-2.6	-4.5	-5.8	-6.8
0.85	64.6	19.7	8.2	3.0	0	-1.9	-3.3	-4.3	-5.0
0.90	37.3	12.3	5.2	2.5	0	-1.3	-2.1	-2.8	-3.3
0.95	16.3	5.8	2.5	0.9	0	-0.6	-1.0	-1.4	-1.6
1.05	-12.9	-5.1	-2.3	-0.9	0	0.6	1.0	1.3	1.6
1.10	-23.1	-9.6	-4.4	-1.7	0	1.2	2.0	2.6	3.1
1.20	-38.1	-17.4	-8.3	-3.2	0	2.2	3.8	5.1	6.1
1.50	-60.6	-33.1	-17.1	-6.9	0	5.0	8.8	11.7	14.1
2.00	-73.5	-46.9	-26.4	-11.3	0	8.7	15.5	21.0	25.5

Table 1: μ in %

Closer investigation of Table 1 yields the following insights: (I) The approximation error decreases as the capital share is closer to 0.5 and (II) as the economy's initial position is closer to its steady state. (III) High[16] capital shares yield smaller approximation errors than low ones do and (IV) the quality of the approximation is higher if capital is overaccumulated.

Consider a model economy with $\{\alpha = 0.3,\ n + \delta = 0.1\}$ and suppose that per capita income is half of its steady state level, i.e. $\lambda_o = 0.5$. The approximated half-life of the initial steady state gap is $\tilde{T}_y = 10$ years and the exact half-life is $T_y = 7$ years leading to a half-life error of 40%. Suppose one could agree to regard linearization errors of the order <10% as totally negligible and orders of 10-20% as basically negligible. With this value judgement in mind, the linearization error is not serious if $\alpha > 0.36$ and the economy's output has at least reached half of its steady state value. Things are quite different for smaller capital shares. The error can only be neglected if the economy has at least reached 70%/80%/95% of its steady state income with a capital share of 0.3/0.2/0.1. Thus the

[16] Here I consider α to be a 'high' capital share if $\alpha > 0.5$. If $\alpha < 0.5$, the capital share is 'low'.

size of the capital share plays a key role when it comes to the performance of $\tilde{\beta}_y$ as a proxy for β_y. Of course, from the preceding table it is more than obvious that the economy's initial position is important, too – but this has been clear from the beginning. In essence we can conclude that $\tilde{\beta}_y$ is a surprising good proxy of the true convergence speed for reasonable paramter values.

Notice that there is a systematic pattern of over- and underestimation of half-lifes: If the economy starts below its long-run equilibrium and the capital share is smaller than 0.5 or it starts above its steady state and α exceeds 0.5, half-life prediction based on linearization always overestimates the true time needed for adjustment. In all other cases, underestimation prevails. The reason for this is the behavior of the transition speed. Figures 3a-c illustrate β_y's behavior as y approaches its steady state. Appealing to proposition 4 shows that its convergence speed either rises ($\alpha > 0.5$), remains constant ($\alpha = 0.5$), or falls ($\alpha < 0.5$) as output per worker increases. For a capital share smaller than 0.5, the transition speed of per capita output declines though the adjustment speed of capital per worker rises as y and k increase.

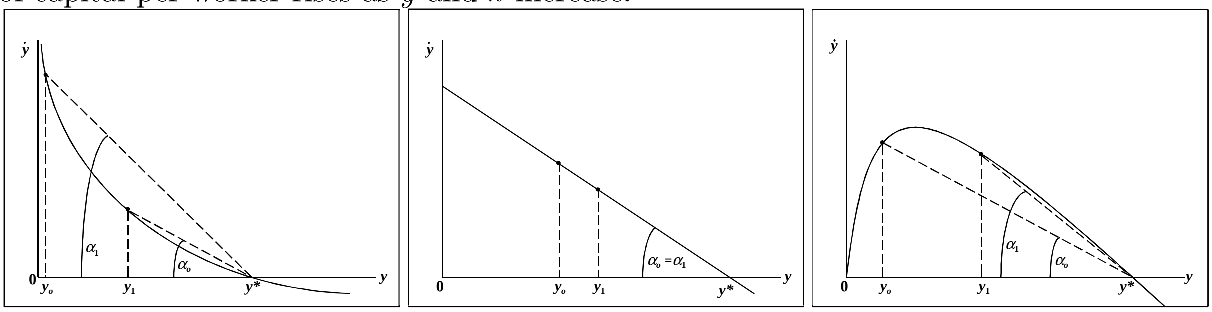

Fig. 3a: $\alpha < 0.5$ Fig. 3b: $\alpha = 0.5$ Fig. 3c: $\alpha > 0.5$

4.2 CES-Production Function

Recent research by Duffy and Papageorgiou (2000) has casted doubt on the conventional choice of the Cobb-Douglas production function as a satisfying description of aggregate output's determination. Specificially they show that for countries with a relatively high capital stock a CES production function with an elasticity of substitution exceeding unity explains the data considered by them better than a Cobb-Douglas-specification.

In light of this it might be interesting to know how the quality of the approximated convergence speed is affected if the elasticity of substitution differs from unity. Consider a CES production function in intensity form given by

$$f(k) = A\left[\alpha k^{-\psi} + (1-\alpha)\right]^{-\frac{1}{\psi}}$$

where the elasticity of substitution is $\sigma = \frac{1}{1+\psi}$ and $-1 < \psi < \infty$. For $\psi = 0$ the CES production function reduces to the Cobb-Douglas-specification. Let δ denote gross depreciation, i.e. it includes capital's "depreciation" due to population growth and technical progress. It is well known that a nontrivial stationary steady state exists if and only if (I) $-1 < \psi < 0$ and $\delta/s > A\alpha^{-\frac{1}{\psi}}$, (II) $\psi = 0$ or (III) $\psi > 0$ and $\delta/s < A\alpha^{-\frac{1}{\psi}}$.[17]

The capital accumulation equation of the Solow model can be written in terms of output, i.e.:

$$\dot{y} = \alpha\delta Ak\left[\frac{sA}{\delta}\left(\alpha + (1-\alpha)k^{\psi}\right)^{-\frac{2+\psi}{\psi}} - \left(\alpha + (1-\alpha)k^{\psi}\right)^{-\frac{1+\psi}{\psi}}\right] \tag{9}$$

where $k = f^{-1} = \left[\frac{\left(\frac{A}{y}\right)^{-\psi} - (1-\alpha)}{\alpha}\right]^{-\frac{1}{\psi}}$. Linearization of $\dot{y}$ around its steady state $y^* = \left[\frac{A^{\psi} - \alpha\left(\frac{\delta}{a}\right)^{\psi}}{1-\alpha}\right]^{\frac{1}{\psi}}$ leads to

$$\dot{y} \cong -\tilde{\beta}_y\left(y - y^*\right).$$

where the convergence coefficient is $\tilde{\beta}_y = \delta\left[1 - \alpha\left(\frac{\delta}{sA}\right)^{\psi}\right]$.

In order to evaluate the quality of the convergence speed measure $\tilde{\beta}_y$, I calculate the relative loss measure μ as described in the preceding. Since no analytical solution to (9) is available, numerical methods have to be invoked to find true half-lives. As the benchmark parameter set $\{s = 0.2,\ \delta = 0.06,\ A = 1,\ \psi = -0.1\}$ is used, implying that $\sigma = 1.\bar{1}$. The choice of σ is guided by the results Duffy and Papageorgiou (2000) report for countries with a relatively high capital intensity. They estimate ψ to be around -0.8. Table 2 reports the results for various initial positions λ_o and values of the distribution parameter α.

$\lambda_o \setminus \alpha$	0.1	0.3	0.5	0.7	0.8
0.50	664.9[18]	28.9	-4.1	-14.5	-17.4
0.75	121.0	10.1	-1.9	-6.5	-7.9
0.80	80.4	7.6	-1.6	-5.1	-6.2
0.90	29.1	3.4	-0.8	-2.5	-3.0
1.10	-17.8	-2.8	0.7	2.3	2.8
1.50	-47.0	-10.4	3.3	10.3	12.7

Table 2: μ in %

[17] Cf. Barro/Sala-i-Martin (1995).

[18] This value is calculated for $\lambda_o = 0.53$ because $y_o = y^*/2$ is not feasible due to k's nonnegativity constraint.

From a comparison between tables 1 and 2 it is evident that the choice of α influences μ similarly to the choice of the capital share in the Cobb-Douglas framework. Higher levels of α generally lead to smaller approximation errors than lower ones do. Furthermore the linearization error is much smaller if adjustment from overaccumulation situations takes place. Here it appears that for $\alpha \geq 0.3$ and $\lambda_o \geq 0.5$ the quality of $\tilde{\beta}_y$ seems to be good. However, this result strongly depends on the choice of s, δ, A and in particular on the elasticity of substitution. Table 3 presents the largest relative error μ obtained if the benchmark parameters were varied such that $s \in \{0.05,\ 0.1,\ 0.2,\ 0.25\}$, $\delta \in \{0.02,\ 0.06,\ 0.1,\ 0.15\}$, $\psi \in \{-0.1,\ -0.25,\ -0.5\}$ and $\alpha \in B = \{0.4,\ 0.5,\ 0.6\}$ or $\alpha = 0.3$. Relative steady state-distances were varied between 0.5 and 0.9. Since errors from overaccumulation seem smaller, the table also applies to initial positions exceeding unity.

	$\psi = -0.1$	$\sigma = 1.\bar{1}$	$\psi = -0.25$	$\sigma = 4/3$	$\psi = -0.5$	$\sigma = 2$
λ_o	$\alpha = 0.3$	$\alpha \in B$	$\alpha = 0.3$	$\alpha \in B$	$\alpha = 0.3$	$\alpha \in B$
0.5	59.5	23.4	115.3	45.5	$68.2^{\lambda_o=0.71}$	$59.0^{\lambda_o=0.63}$
0.6	39.2	16.1	63.9	28.2	$68.2^{\lambda_o=0.71}$	$59.0^{\lambda_o=0.63}$
0.75	24.9	10.6	26.3	12.9	45.2	21.2
0.8	14.3	6.3	18.8	9.4	27.4	14.2
0.9	6.3	3.3	7.7	4.0	9.2	5.3

Table 3: Largest $|\mu|$ in % for $\sigma > 1$

Obviously the approximation quality decreases as the elasticity of substitution increases. For parameter variations with $\psi = -0.1$, the approximation quality is still good if α exceeds 0.3. For still larger elasticities of substitution either endogenous growth prevails or the error's magnitude is small, the latter partly due to initial steady state vicinity.

A different picture emerges if the elasticity of substitution falls below unity. To highlight this, consider the benchmark parameter set with $\alpha = 0.3$ and $\lambda_o = 0.75$ for different elasticities of substitution. The results are summarized in table 4.

σ	2	4/3	$1.\bar{1}$	$0.\bar{9}\bar{0}$	2/3	0.5
μ in %	-0.5	4.6	10.1	21.1	57.2	128.4
Half-Life	25.7	18.6	15.9	13.0	8.8	5.6
$\tilde{\beta}_y$	.027	.036	.040	.044	.050	.055

Table 4

For a low degree of substitutability between capital and labor, i.e. $\sigma < 1$, the performance of the approximated convergence speed is low. It dramatically worsens as substitutability is further reduced. Notice that the figures apply for a steady state gap which is a quarter of the steady state level. If the scale parameter A increases, the given figures for low elasticities of substitution worsens. The data contained in table 5 reinforces the claim that the quality of $\tilde{\beta}_y$ is low if low substitutability prevails. In the spirit of table 3 it gives the largest loss due to linearization if benchmark parameters are varied as before.

	$\psi = 0.1$	$\sigma = 0.\bar{9}\bar{0}$	$\psi = 0.5$	$\sigma = 2/3$	$\psi = 1$	$\sigma = 1/2$
λ_o	$\alpha = 0.3$	$\alpha \in B$	$\alpha = 0.3$	$\alpha \in B$	$\alpha = 0.3$	$\alpha \in B$
0.5	74.4	33.8	352.5	218.5	875.2	642.2
0.6	53.1	25.0	272.4	172.2	783.1	575.6
0.75	28.4	14.1	158.2	103.2	566.3	417.1
0.8	21.7	10.9	122.8	81.1	473.5	349.0
0.9	9.9	5.1	57.3	38.8	257.9	190.3

Table 5: Largest $|\mu|$ in % for $\sigma > 1$

From the Cobb-Douglas-specification and the results for the CES-production function it can be concluded that, for reasonable parameters, the quality of the approximated transition speed is quite good as long as the the elasticity of substitution is negligibly below unity. If the degree of substitutability between capital and labor is low, $\tilde{\beta}_y$ is not a good measure of the convergence speed.

5 Application to a Ramsey Model

Consider a standard Ramsey model as discussed in Barro/Sala-i-Martin (1995). In order to make the following results comparable to Ortigueira/Santos (1997), let the consumption function be of CIES type, the production function be Cobb-Douglas with scale parameter A and ignore technical progress. Maximizing the social planner's problem yields an unique solution $\{k(t), c(t)\}$ which is characterized by the differential equation system:

$$\begin{aligned} \dot{k}(t) &= k(t)^{\alpha} - c(t) - (n+\delta)k(t) \\ \dot{c}(t) &= \frac{c(t)}{\theta}\left[\alpha k(t)^{\alpha-1} - (\delta+\rho)\right]. \end{aligned} \tag{10}$$

The notation is standard, particularly c and k denote consumption and capital per worker respectively. The rate of labor force growth is given by n, the depreciation rate of capital is δ, and α is the capital share. Households' subjective time preference rate is ρ and $1/\theta$ gives the intertemporal elasticity of substitution. Setting $\dot{k} = \dot{c} = 0$ implies the steady state values

$$k^* = \left(\frac{\alpha}{\delta + \rho}\right)^{\frac{1}{1-\alpha}}, \qquad c^* = k^{*^{\alpha}} - (n+\delta)k^*.$$

In order to find the approximated convergence coefficient for per capita output, system (10) has to be linearized around $\{c^*, k^*\}$. If $\tilde{k}(t)$ is known, then per capita output's approximated time path can be written as[19]

$$\tilde{y}(t) = (y_o - y^*)e^{-\tilde{\beta}_y t} + y^*, \tag{11}$$

where $\tilde{\beta}_y$ is the approximated convergence coefficient equal to the negated negative eigenvalue of the differential equation system:

$$\tilde{\beta}_y = \sqrt{\frac{(\rho - n)^2}{4} + \frac{1-\alpha}{\theta}(\delta + \rho)\frac{(1-\alpha)\delta + \rho - \alpha n}{\alpha}} - \frac{\rho - n}{2}.$$

Since the convergence coefficient's proxy is a constant, the approximated half-life is given by

$$\tilde{T}_y = \frac{\ln 2}{\tilde{\beta}_y}.$$

It is more complicated to obtain the (numerical) true half-life. Here, the time-elimination method is used to derive the saddle path $c(k)$. Next, vectors c and k are used to determine $\dot{k}$ according to its equation of motion (10). Using this information and the fact that $\dot{y} = \alpha k^{\alpha-1}\dot{k}$ yields y's time derivative. Together with y, half-lives can be calculated.

The parameter set $\{n = 0.01,\ \delta = \rho = 0.05,\ \theta = 1.5\}$ is used as the benchmark case. It has also been used by Ortigueira/Santos (1997) and Lucas (1988) among others. The capital share is varied between 0.1 and 0.9 and different relative initial positions of y are used. Table 2 summerizes relative losses μ:

$\lambda_o \backslash \alpha$	0.1	0.3	0.5	0.7	0.9
0.5	1375.8	56.9	3.4	-12.4	-19.2
0.7	334.2	26.5	1.0	–7.1	–12.0
0.9	54.9	7.1	0.3	-3.2	-4.0

Table 2: Relative loss μ in % of true half-life

[19] See the appendix for the full derivation.

Surprisingly the results contained in that table are very robust for all analyzed parameter sets. Moreover, all results obtained in the Solow framework, even the error's magnitude, are carried over: α is a key parameter as well as the economy's initial position. All other parameters seem to have no strong effect on the error due to linearization.

6 Conclusion

This paper has argued that is is a priori not clear whether the linearization error due to first-order approximations of nonlinear dynamic systems is negligible if convergence speed issues are discussed. A method was developed which allows to quantify the quality of approximations based on linearization. Its application to neoclassical growth theory with a Cobb-Douglas production function shows that the capital share is a key parameter: adjustement times based on the approximated convergence coefficient for per capita income are over- or underestimated compared to true times depending on $\alpha \gtrless 0.5$. If $\alpha = 0.5$ the linearization is a perfect approximation: no error arises. For capital shares which are not too small, the error due to linearization seems to be minor. However, if the elasticity of substitution is below unity, the approximation procedure yields unreliable results.

7 Appendix

7.1 Derivation of (6a) and (7)

T_y — Define $\phi \equiv y^{\frac{1-\alpha}{\alpha}}$ and transform (5) into a linear differential equation of the form:

$$\dot{\phi} = (1-\alpha)(n+\delta)\left(y^{*\frac{1-\alpha}{\alpha}} - \phi\right).$$

After solving that equation re-change variables to obtain the exact time path for output per worker

$$y(t) = \left\{\left(y_o^{\frac{1-\alpha}{\alpha}} - y^{*\frac{1-\alpha}{\alpha}}\right) e^{-(1-\alpha)(n+\delta)t} + y^{*\frac{1-\alpha}{\alpha}}\right\}^{\frac{\alpha}{1-\alpha}}.$$

Inverting the above given equation and using λ's definition yields[20]

$$t(\lambda) = \frac{\ln\left[d\left(1-\lambda^{\frac{1-\alpha}{\alpha}}\right)\right] - \ln\left\{d\left[1-\left(\frac{1+\lambda_o}{2}\right)^{\frac{1-\alpha}{\alpha}}\right]\right\}}{(1-\alpha)(n+\delta)}, \qquad d = \begin{cases} 1 & \text{f. } \lambda, \lambda_o \in (0,1) \\ -1 & \text{f. } \lambda, \lambda_o > 1 \end{cases}.$$

[20] The dummy variable d controls the sign of both log-arguments such that those arguments are strictly positive.

According to the half-life condition, T_y is determined by $y(T) = \frac{y^*+y_o}{2}$. Plugging this argument into the above given equation yields (6a).

$\tilde{T}_y$ — Equation (2) has been obtained by linear approximation around y's steady state. Solving that approximation for the associated time path of per capita income, $\tilde{y}(t)$, leads to:

$$\tilde{y}(t) = (y_o - y^*)e^{-(1-\alpha)(n+\delta)t} + y^*.$$

The inverse is given by

$$\tilde{t}(\lambda) = \frac{\ln\left[d(1-\lambda_o)\right] - \ln\left[d(1-\lambda)\right]}{(1-\alpha)(n+\delta)}, \qquad d = \begin{cases} 1 & \text{f. } \lambda, \lambda_o \in (0,1) \\ -1 & \text{f. } \lambda, \lambda_o > 1 \end{cases}.$$

Plugging the half-life condition written in terms of λ, i.e. $\lambda(T) = \frac{1+\lambda_o}{2}$, into above yields the proposed result:

$$\tilde{T}_y = \frac{\ln 2}{(1-\alpha)(n+\delta)}.$$

7.2 Derivation of (11)

The linearized differential equation system may be written as

$$\begin{aligned} \dot{k}(t) &= (\rho - n)\, k(t) - c(t) \quad - (\rho - n)\cdot k^* + c^*, \\ \dot{c}(t) &= c^*\theta^{-1}\alpha(\alpha-1){k^*}^{\alpha-2}k(t) - \frac{c^*}{\theta}\alpha(\alpha-1){k^*}^{\alpha-1}. \end{aligned} \tag{12}$$

That system's characteristic roots are given by

$$\lambda_{1,2} = \frac{\rho - n}{2} \pm \sqrt{\frac{(\rho-n)^2}{4} + \frac{c^*}{\theta}\alpha(1-\alpha){k^*}^{\alpha-2}},$$

Solution of that system under Ramsey conditions yields capital's approximated time path:

$$\tilde{k}(t) = (k_o - k^*)e^{-\beta t} + k^*, \tag{13}$$

where $\beta = -\lambda_2$. From the approximated path $\tilde{k}(t)$, we get $\dot{k}(t) = -\beta(k_o - k^*)e^{-\beta t}$ and $(k_o - k^*)e^{-\beta t} = k(t) - k^*$. Substituting the latter into the former yields

$$\dot{k}(t) = -\beta(k(t) - k^*) \tag{14}$$

If we pursue a first-order Taylor approximation of the production function in intensive form, $y(t) = k(t)^\alpha$, around the steady state, we obtain

$$y(t) - y^* = \alpha {k^*}^{\alpha-1}(k(t) - k^*) \tag{15}$$

Differentiation of $y(t)$ with respect to time gives $\dot{y}(t) = \alpha k(t)^{\alpha-1}\dot{k}(t)$. A first-order Taylor expansion around the steady state of $\dot{y}(t)$ using $d\dot{k}/dk = -\beta$ from (14) leads to

$$\dot{y}(t) = -\beta\alpha {k^*}^{\alpha-1}(k(t) - k^*).$$

Substitution of (15) into that expression yields

$$\dot{y}(t) = -\beta\,(y(t) - y^*)$$

whose solution is (11).

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