Optimal Auctions with Outside Competition

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Abstract

This paper studies how a revenue-maximizing auction seller responds to competition outside the auction. Outside competition is modeled by buyer-sided outside options. Since outside options for buyers increase the degree of competition from the seller's point of view, intuition suggests that a revenue-maximizing seller might seek to enhance the competitiveness of her auction offer. In contrast, it is shown that the optimal response to outside options calls for a less competitive auction as measured by the probability of a sale. For the first-price and second-price auction, it is shown how the optimal minimum bid varies with the level of competition.

1 Introduction

This paper introduces outside competition into the Symmetric Independent Private Values Model (SIPV). Specifically competition outside the auction is assumed to be exogenous and modeled as a buyer-sided outside option following Kirchkamp, Poen and Reiss (2009). Buyer-sided outside options constitute mutually exclusive consumption opportunities to the object auctioned off and there is no buyer-competition for outside options. From the seller's point of view, additional consumption opportunities of buyers increase the degree of competition. Uneducated common sense suggests that a revenue-maximizing seller might seek to enhance the competitiveness of her auction offer as a response. In contrast, it is shown that optimal auctions with outside options are less competitive as measured by the probability of a sale although the optimal minimum bid in the first-price and second-price auction decreases with the level of competition as proxied by the value of outside options to buyers.

The plan of this note is as follows: section 2 reviews the SIPV model with public outside options under the assumption of risk neutrality. Section 3 characterizes optimal auctions and discusses the result. Section 4 considers the behavior of optimal minimum bids in the first-price and second-price auction. Section 5 briefly comments on allocative efficiency.

2 The SIPV model with public outside options

In this section, the standard SIPV auction model with public outside options that are buyer-sided is reviewed.[1] First, the standard framework is reviewed. Then, public outside options are discussed. Subsequently, equivalence results are provided serving as the foundation to the characterization of seller-optimal auction designs.

2.1 The standard model

Consider the canonical auction model with independent private values where a single seller allocates an indivisible object to a group of $N > 1$ risk-neutral potential buyers employing some auction design of her choice.[2] The seller's reservation value for the object is $v_o \in \mathbb{R}$. Buyer i ($i = 1, ..., N$) values the object at $v_i \in [v_-, v_+]$ which is his private information, $[v_-, v_+] \subset \mathbb{R}$. Bidders' valuations are independently and identically distributed according to the cumulative distribution function $F(v_i)$ which is continuously differentiable and strictly monotonic increasing on its domain $[v_-, v_+]$.

[1]See Kirchkamp, Poen and Reiss (2009) for a more detailed discussion.

[2]The model as outlined was introduced in Riley and Samuelson (1981).

The seller is restricted to choose an auction design that conforms to the following assumptions A1-A4:

A1 The seller always accepts bidder i's bid $b_i \in \mathbb{R}$ unless it is smaller than the minimum bid $b_M \in \mathbb{R}$.

A2 The auction winner is always a bidder with the highest bid.

A3 The auction design only discriminates between bidders on the basis of submitted bids.

A4 There exists a symmetric Bayesian equilibrium characterized by a strictly monotonic increasing bidding function $\beta : [v_-, v_+] \to \mathbb{R}$ and a participation rule $\varepsilon : [v_-, v_+] \to \{\textit{bid submission}, \textit{no bid submission}\}$ where $v_{\sim}$ is the lowest valuation of a bidder that voluntarily participates in the auction such that $\varepsilon(v_i) = \textit{no bid submission if } v_i < v_{\sim}$ and $\varepsilon(v_i) = \textit{bid submission if } v_i \geq v_{\sim}$. If each valuation type $v \in [v_-, v_+]$ voluntarily participates in the auction, then $v_{\sim} = v_-$, otherwise $v_{\sim}$ denotes the valuation of a bidder who is indifferent between bid submission and nonparticipating in the auction.

Assumptions A1-A3 ensure that the auction design exhibits bidder anonymity and excludes bidder-specific side payments unless these are formed on the basis of bids only. Assumption A2 allows for any tie-breaking rule; its particular specification is meaningless since, due to A4, coincidence of bids requires coincidence of continuously distributed valuations which occurs with probability zero.

Definition 1 *Auction design $\mathcal{D}$ belongs to the class of auctions $\mathcal{A}$ if it conforms to assumptions A1-A4.*

The model as outlined and the particular auction design is common knowledge.

2.2 Public outside options

The novel feature of the studied auction model is the introduction of buyer-sided consumption opportunities. In addition to the consumption opportunity associated with winning the auctioned object, here, each of the N buyers has the opportunity to substitute the auctioned object by some certain alternative. Without loss of generality, only the best alternative is considered if there is more than one. A potential bidder can execute his outside option instead of participating in the auction or subsequently to his auction participation if he did not win the object.[3]

In particular, if bidder i executes his outside option, he receives the certain payoff $u(w_i)$ where $w_i \in [w_-, w_+] \subset \mathbb{R}$. To ensure existence of valuation types that have an incentive to

[3] This feature distinguishes the studied type of outside options from that in the common value model in Cox, Dinkin and Swarthouth (2001) where mere auction participation forfeits the outside option's value.

participate in a standard auction[4], it is assumed that the largest valuation v_+ always exceeds the value of the outside option w_i:

$$v_+ > w_i. \tag{1}$$

Otherwise, even at a zero-cost, buyers prefer their outside options to the auctioned object and the considered auction problem is trivial and meaningless.

In order to formalize that the auctioned object and the outside option constitute mutually exclusive alternatives, suppose in the extreme that receiving the auction object fully forfeits the value of the outside option. Thus, single-object demand is assumed. Moreover, suppose that outside options are symmetric and public in the sense that all outside option values coincide for all bidders which is common knowledge:

$$\omega_i = \begin{cases} w_i, & i \text{ without auctioned object} \\ 0, & i \text{ with auctioned object} \end{cases}$$
$$\text{where: } w_i = w_j = w, \quad (\forall i, j = 1, ..., N).$$

In a symmetric Bayesian equilibrium characterized by a strictly monotonic increasing bidding schedule, the "lowest" valuation type participating in the auction never wins it and always realizes his outside option. Thus, the lowest valuation type that voluntarily participates in the auction requires in the auction:

$$U(\beta(v_{\sim}), v_{\sim}) = u(w). \tag{2}$$

In case there is no outside option, $w = 0$ and $U(\beta(v_{\sim}), v_{\sim}) = u(0)$ imply the canonical SIPV model as a special case.

2.3 Equivalence theorems

This subsection provides equivalence theorems under risk-neutrality that allow for public outside options. For completeness, the theorems are explicitly derived using the well-known standard procedure. Firstly, it is shown that expected payments of buyers are independent of the particular auction design as long as the pool of participating valuation types remains unchanged. Secondly, it is shown that bidders' expected payment equivalence implies expected payoff equivalence for buyers and sellers.[5] All equivalence theorems in the absence of outside options were independently developed in Riley and Samuelson (1981) and Myerson (1981).[6]

[4] Without entry fees and strictly positive minimum bids etc.

[5] Seller's payoff equivalence is also known as revenue equivalence.

[6] Surveys are provided in Klemperer (1999), McAfee and McMillan (1987), Milgrom (1989, 2004) and Wolfstetter (1996).

The introduction of public outside options does not essentially alter the equivalence results and leads to the intuitive insight that expected equilibrium payments of buyers decrease with the outside option's value. It follows that more valuable outside options lead to a larger payoff to a potential buyer and to a smaller revenue for the seller. In this sense, outside options lead to a transfer of rents from the seller to the group of buyers. The presentation of results follows Riley and Samuelson (1981) and Wolfstetter (1996) that ignore outside options.

2.3.1 Expected payment equivalence

Risk-neutrality implies a positive affine Bernoulli utility function that is employed in its normalized form $u(x) = x$. Here, the property that risk neutrality allows additive separation of expected benefits and expected costs is important. If the actual payment rule under any auction design of class $\mathcal{A}$ is denoted by $z(b_1, b_2, ..., b_N)$, the expected payment of any bidder i submitting bid b_i can be written as[7]

$$\begin{aligned} U(b_i, v_i) = &\Pr\{b_i = \max\{\beta(V_1), ..., b_i, ..., \beta(V_N)\}\} \cdot v_i - E\left[z(\beta(V_1), ..., b_i, ..., \beta(V_N))\right] \\ &+ \left[1 - \Pr\{b_i = \max\{\beta(V_1), ..., b_i, ..., \beta(V_N)\}\}\right] \cdot w, \end{aligned}$$

where the unknown valuation of any bidder $j \neq i$ is given by V_j and all other bidders are supposed to follow the assumed equilibrium bidding strategy $\beta(\cdot)$. Simple algebraic manipulation of the preceding definition leads to

$$\begin{aligned} U(b_i, v_i) = &\Pr\{b_i = \max\{\beta(V_1), ..., b_i, ..., \beta(V_N)\}\} \cdot \underbrace{(v_i - w)}_{\text{net valuation}} \\ &- E\left[z(\beta(V_1), ..., b_i, ..., \beta(V_N))\right] + w. \end{aligned} \tag{3}$$

where the net valuation is defined as the difference of a bidder's valuation for the auctioned object and the value of his outside option. The last constant signals that the bidder can be sure to receive his outside option. The first term shows that, in case the bidder wins the auction, he receives his net valuation: he does receive his valuation for the auctioned object but has to "repay" the value of his outside option at the same time.

A central result in this model is the broad independence of (valuation-dependent) expected buyer payments of any particular auction design. This result is known as payment equivalence and stated as theorem **??**. It holds for all auction designs of class $\mathcal{A}$ that leave the participating valuation pool unchanged. It is obvious that independence of expected payments can not hold if the valuation pool changes with the auction design. Suppose it did, then, there must be some valuation type that makes nonzero payments under one design but zero payments under some

[7] To obtain bidder i's winning probability, notice that the assumed equilibrium bidding function $\beta(\cdot)$ is symmetric and strictly monotonic increasing.

other design directly contradicting design-independence of expected payments for this type. Moreover, it will be shown that a revenue-maximizing seller restricts the participating valuation pool which only increases revenue if some (participating) valuation types make larger expected payments with valuation type restriction than without since the increased expected revenue is generated by a smaller pool of valuation types.

Theorem 1 (payment equivalence) *For any auction design in class $\mathcal{A}$ and in a symmetric Bayesian equilibrium, the expected equilibrium payment to the seller $Z^*(v_i) \equiv E[z(\beta(V_1), ...,\beta(v_i),..., \beta(V_N))]$ by bidder i with $v_i \geq v\text{\textasciitilde}$ is:*

$$Z^*(v_i) = v_i\, F(v_i)^{N-1} - \int_{v\text{\textasciitilde}}^{v_i} F(x)^{N-1}\, dx - w\, F(v_i)^{N-1}. \tag{4}$$

Proof. [Sketch]

The objective function (**??**) implies:

$$\frac{dU(b_i, v_i)}{dv_i} = \frac{\partial U(b_i, v_i)}{\partial b_i} \cdot \frac{\partial b_i}{\partial v_i} + \frac{\partial U(b_i, v_i)}{\partial v_i}.$$

Optimal bidding, $b_i^*(v_i)$, requires $\partial U(b_i^*(v_i), v_i)/\partial b_i = 0$ and thus:

$$\frac{dU(b_i^*, v_i)}{dv_i} = \frac{\partial U(b_i^*, v_i)}{\partial v_i}.$$

Substitution of the partial derivative $\partial U/\partial v_i$ and using definition $U^*(v_i) \equiv U(b_i^*(v_i), v_i)$ leads to:

$$\frac{dU^*(v_i)}{dv_i} = \Pr\left\{b_i^* = \max\left\{\beta(V_1),\, ...,\, b_i^*, ..., \beta(V_N)\right\}\right\}.$$

Since the equilibrium is symmetric by assumption and due to its Nash-property, there must be no incentive for bidder *i* to deviate from the equilibrium strategy implying $b_i^*(v_i) = \beta(v_i)$ and thus:

$$\frac{dU^*(v_i)}{dv_i} = \Pr\left\{\beta(v_i) = \max\left\{\beta(V_1),\, ...,\, \beta(v_i), ..., \beta(V_N)\right\}\right\}.$$

Since the equilibrium bidding function is strictly monotonic increasing, the winning probability is given by:

$$\begin{aligned}
&\Pr\left\{\beta(v_i) = \max\left\{\beta(V_1),\, ...,\, \beta(v_i), ..., \beta(V_N)\right\}\right\} \\
&= \Pr\left\{v_i = \max\left\{V_1,\, ...,\, v_i, ..., V_N\right\}\right\} \\
&= F(v_i)^{N-1},
\end{aligned}$$

where the last line follows from the assumption of independent and identical distributions of other bidders' valuations. It follows that

$$\frac{dU^*(v_i)}{dv_i} = F(v_i)^{N-1}.$$

Integrating over $[v_{\sim}, v_i]$ leads to

$$U^*(v_i) = U^*(v_{\sim}) + \int_{v_{\sim}}^{v_i} F(x)^{N-1}\, dx. \tag{5}$$

Due to the objective function's definition (**??**), in equilibrium:

$$U^*(\, v_i) = F(v_i)^{N-1} \cdot (v_i - w) - E\left[z(\beta(V_1), ..., \beta(v_i), ..., \, \beta(V_N))\right] + w. \tag{6}$$

Equating both expressions for expected utility of bidder i in equilibrium, (**??**) and (**??**), and solving for $E[z(\cdot)]$ leads to:

$$E\left[z(\beta(V_1), ..., \beta(v_i), ..., \, \beta(V_N))\right] = F(v_i)^{N-1} \cdot (v_i - w) - \int_{v_{\sim}}^{v_i} F(x)^{N-1}\, dx - U^*(v_{\sim}) + w.$$

Public outside options imply that valuation type $v_{\sim}$ precisely receives from participating in the auction the expected utility that he obtains from seizing his outside option, therefore:

$$U^*(v_{\sim}) = w.$$

It follows that the expected equilibrium payment of bidder i is:

$$E\left[z(\beta(V_1), ..., \beta(v_i), ..., \, \beta(V_N))\right] = F(v_i)^{N-1} \cdot (v_i - w) - \int_{v_{\sim}}^{v_i} F(x)^{N-1}\, dx.$$

□

Example 1 (UD) *Valuations of any potential buyer are uniformly distributed over* $[v_-, v_+]$ *such that the cumulative distribution function is given by:*

$$F(v_i) = \begin{cases} \frac{v_i - v_-}{v_+ - v_-}, & \text{if } v_i \in [v_-, v_+] \\ 0, & \text{otherwise} \end{cases}.$$

The resulting expected equilibrium payment of a potential buyer with valuation v_i *to the auctioneer is:*

$$Z^*(v_i) = \begin{cases} \frac{N\,(v_i - w)\,(v_i - v_-)^{N-1} - (v_i - v_-)^N + (v_{\sim} - v_-)^N}{N\,(v_+ - v_-)^{N-1}}, & \text{if } v_i \in [v_{\sim}, v_+] \\ 0, & \text{otherwise} \end{cases}.$$

Example 2 (ED) *Valuations of two potential buyers are exponentially distributed over* $[0, \infty)$ *depending on parameter* $\lambda > 0$ $(E[V_i] = 1/\lambda)$ *such that the cumulative distribution function is given by:*

$$F(v_i) = \begin{cases} 1 - e^{-\lambda v_i}, & \text{if } v_i \in [0, \infty) \\ 0, & \text{otherwise} \end{cases}.$$

The resulting expected equilibrium payment of a potential buyer with valuation v_i *to the auctioneer is:*

$$Z^*(v_i) = \begin{cases} v_{\sim} + \frac{e^{-\lambda v_{\sim}}}{\lambda} - \left(v_i - w + \frac{1}{\lambda}\right) e^{-\lambda v_i} - w, & \text{if } v_i \in [v_{\sim}, \infty) \\ 0, & \text{otherwise} \end{cases}.$$

2.3.2 Effect of outside options on expected payments

The explicit introduction of public outside options into the SIPV model shows that the outside option value w determines the expected equilibrium payment $Z^*(v_i)$ of a potential buyer. Corollary **??** summarizes that the seller receives lower expected payments for more valuable outside options. If a change in w does not affect the marginal bidder, then the expected payment decreases by $w\,F(v_i)^{N-1}$ as compared to the standard case without outside options ($w = 0$). This payment reduction matches the expected opportunity costs of participating in the auction created by outside options: Bidder i with v_i has the largest valuation among N buyers and wins the auction with probability $F(v_i)^{N-1}$ ($\beta' > 0$). With this winning probability, he forfeits his outside option valued at w at the same time, for which equilibrium behavior exactly compensates him.

Corollary 1 *More valuable public outside options reduce the expected equilibrium payment of any bidder $v_i > \tilde{v}$ unless the response of the marginal bidder $\tilde{v}$ fully compensates this in cases with restricted valuation pool, i.e. $\tilde{v} > v_-$.*

Proof. Differentiating $Z^*(v_i)$ leads to

$$\frac{\partial Z^*(v_i)}{\partial w} = -F(v_i)^{N-1} + \frac{\partial \tilde{v}}{\partial w}\, F(\tilde{v})^{N-1}.$$

By definition $F(v_-) = 0$ and the second term of the derivative vanishes for $\tilde{v} = v_-$. It follows that the expected equilibrium payment of any bidder $v_i > v_-$ decreases in w. For $\tilde{v} > v_-$, the response of the marginal bidder to variations of w, $\partial \tilde{v}/\partial w$, influences the sign of $\partial Z^*/\partial w$, too. Since F is strictly monotonic increasing, for $v_i > \tilde{v}$ one always obtains $F(v_i)^{N-1} > F(\tilde{v})^{N-1}$, thus, $\partial \tilde{v}/\partial w \leq 1$ is sufficient for a negative response of $\partial Z^*/\partial w$.

□

For standard auction designs such as the first-price or the second-price auction where the marginal bidder is implemented by a minimum bid, the marginal bidder corresponds to the sum $b_{\mathrm{M}} + w$ implying $\partial \tilde{v}/\partial w = 1$ and, thus, for these designs $\partial Z^*/\partial w < 0$.

2.3.3 Equivalence of expected payoffs

A consequence of the fundamental payment equivalence theorem is its straightforward extension to expected equilibrium payoffs: Since the expected payments of bidders coincide for all designs belonging to class $\mathcal{A}$ under invariance of the valuation pool, its sum, which is expected revenue, must remain unchanged. Due to the fact that the probability of winning solely depends on realized valuations that remain constant as the design changes, it is obvious that the expected payoff to any valuation type under any design of class $\mathcal{A}$ must coincide. The following theorem **??** formalizes these considerations. From a bidder's perspective, the introduction of public outside options leads equilibrium behavior to imply an increase in his value function by w although the Bernoulli utility function does not increase by w. This demonstrates that the existence of outside options leaves a (nonoptimizing) seller worse off since equilibrium bidding behavior ensures that each bidder receives at least his outside option value. If a bidder does not win the auction, he receives his outside option without imposing any cost on the seller. However, winning the auction forfeits the winner's outside option and his virtual execution of the outside option, guaranteed by bidding behavior, requires the seller to provide the outside options value through a lower payment for the auctioned object. Due to that, expected revenue drops by the value of the public outside option given that at least one bidder participates in the auction occurring with probability $1 - F(v_-)^N$.

Theorem 2 (payoff equivalence) *For any auction design in class $\mathcal{A}$ and in a symmetric Bayesian equilibrium,*
(a) (revenue equivalence) the expected revenue $R^*(v_-)$ of the seller is given by

$$R^*(v_-) = N \int_{v_-}^{v_+} F(v_i)^{N-1} f(v_i) \left[v_i - \frac{1 - F(v_i)}{f(v_i)} \right] dv_i - w \left[1 - F(v_-)^N \right], \tag{7}$$

(b) (payoff equivalence/buyer) and the expected payoff of buyer i (i=1, ..., N) is

$$U^*(v_i) = \begin{cases} \int_{v_-}^{v_i} F(x)^{N-1}\, dx + w, & \text{if } v_i \geq v_- \\ w, & \text{if } v_i < v_- \end{cases} . \tag{8}$$

Proof. (a) Each bidder with valuation $v_i \geq v_-$ expects to pay $Z^*(v_i)$ to the seller in symmetric Bayesian equilibrium (theorem **??**). The seller receives no expected payment from any potential bidder with valuations smaller than the indifference threshold v_-. Since the seller is not informed about valuation realizations, she expects to receive the same payment from each of the N potential bidders:

$$\begin{aligned} R^*(v_-) &= N \cdot [\Pr\{v_i \geq v_-\} \cdot E[Z^*(v_i) | v_i \geq v_-] + \Pr\{v_i < v_-\} \cdot 0] \\ &= N \int_{v_-}^{v_+} f(v_i)\, Z^*(v_i)\, dv_i \\ &= N \int_{v_-}^{v_+} f(v_i)\, v_i\, F(v_i)^{N-1} dv_i - N \underbrace{\int_{v_-}^{v_+} f(v_i) \int_{v_-}^{v_i} F(x)^{N-1} dx\, dv_i}_{=:\,\mathrm{A}} - N \underbrace{\int_{v_-}^{v_+} f(v_i)\, w\, F(v_i)^{N-1} dv_i}_{=:\,\mathrm{B}} \end{aligned}$$

Integrating by parts expression A leads to[8]

$$\mathrm{A} = \int_{\tilde{v}}^{v_+} F(v_i)^{N-1} dv_i - \int_{\tilde{v}}^{v_+} F(v_i)^N dv_i. \tag{9}$$

Since w is constant and $f(v_i)$ is the derivative of $F(v_i)$ with property $F(v_+) = 1$, the solution of B is given by

$$\mathrm{B} = \frac{w}{N}\left[1 - F(\tilde{v})^N\right]. \tag{10}$$

Using (**??**) and (**??**) leads to the expression for expected revenue

$$R^*(\tilde{v}) = N\int_{\tilde{v}}^{v_+} f(v_i)\, v_i\, F(v_i)^{N-1} dv_i - N\int_{\tilde{v}}^{v_+} F(v_i)^{N-1} dv_i + N\int_{\tilde{v}}^{v_+} F(v_i)^N dv_i - w\left[1 - F(\tilde{v})^N\right].$$

Grouping integrals leads to the expression in part (a) of the theorem.
(b) Each bidder with valuation $v_i < \tilde{v}$ does not participate in the auction and executes his outside option. Thus, his expected utility is $U^*(v_i | v_i < \tilde{v}) = w$. The expected utility of a bidder with valuation $v_i \geq \tilde{v}$ is given by (**??**) depending on $U^*(\tilde{v})$. Using condition $U^*(\tilde{v}) = w$ leads to (b).

□

Example 3 (UD) *Valuations of buyers are uniformly distributed over* $[v_-, v_+]$: $F(v_i) = \frac{v_i - v_-}{v_+ - v_-}$. *Theorem* **??** *implies*

$$R^*(\tilde{v}) = \frac{(N-1)v_+ + 2v_-}{N+1} + \frac{(\tilde{v} - v_-)^N\left[(N+1)(v_+ + w) - 2(N\tilde{v} + v_-)\right]}{(v_+ - v_-)^N (N+1)} - w$$

and

$$U^*(v_i) = \begin{cases} \frac{(v_i - v_-)^N - (\tilde{v} - v_-)^N}{N\,(v_+ - v_-)^{N-1}} + w, & \text{if } v_i \in [\tilde{v}, v_+] \\ w, & \text{otherwise.} \end{cases}.$$

3 Optimal auctions and the decision to offer the auction

In this section, auction designs are identified that maximize the expected payoff to a risk-neutral seller. The presented characterization of optimal auction designs follows the approach of Riley and Samuelson (1981) employing the revenue equivalence theorem. Here, the key result is that the revenue maximizing seller reduces the competitiveness of her auction offer in the sense that the allocation probability decreases as the value of outside options increases.

The revenue equivalence theorem **??** implies that the only way a seller can affect her expected equilibrium revenue is through adjustments of the marginal bidder. Thus, independently of the specific value of the outside option, a change of the auction design that leaves the

[8]The separation of the original integral into $u'(v_i) := f(v_i)$ and $v'(v_i) := \int_{\tilde{v}}^{v_i} F(x)^{N-1}dx$ implies $u(v_i) = F(v_i)$ and application of the Leibniz rule leads to $v'(v_i) = F(v_i)^{N-1}$. Thus, $\int_{\tilde{v}}^{v_+} u'(v_i)\, v(v_i)\, dv_i = \left[F(v_i) \int_{\tilde{v}}^{v_i} F(x)^{N-1}dx\right]_{\tilde{v}}^{v_+} - \int_{\tilde{v}}^{v_+} F(v_i)^N\, dv_i$. Using $F(v_+) = 1$ and $\int_{\tilde{v}}^{\tilde{v}} F(x)^{N-1}dx = 0$, the solution of A is obtained.

participating valuation pool unaffected, e.g. a second-price auction with the same minimum bid and entry fee instead of a first-price auction, does not affect expected revenue in equilibrium. This property significantly eases the characterization of revenue-maximizing auction designs since all that is required for characterization is the optimal selection of the marginal bidder.

In case it is suboptimal for a potential seller to offer an auction, the optimal auction design restricts the participating valuation pool such that the allocation probability is equal to zero. Therefore, optimal auction designs simultaneously solve the seller's decision to offer the auction.

Generality is the main advantage of the identification of optimal auctions through optimal marginal bidder selection using revenue equivalence. Instead of finding the optimal marginal bidder for each auction design which requires knowledge of every specific rule of the considered design, it suffices that the design conforms to some basic assumptions (A1-A4) such that there is auction class-wide optimization: Each auction design in class $\mathcal{A}$ is optimal unless it implies a marginal bidder that is suboptimal. Unfortunately, a revenue maximizing seller cannot directly implement this characterization since the complete set of rules of an auction design implies the particular marginal bidder, thus, the marginal bidder is no control variable of the seller. To address this shortcoming, section 4 demonstrates how the marginal bidder can be implemented by a minimum bid in the first-price and second-price auction.

In order to ease the exposition, it is assumed that the distribution exhibits the *Monotone Hazard Rate Property* given below:[9]

(Monotone Hazard Rate Property) A cumulative distribution function $F(v)$ exhibits the monotone hazard rate property if it satisfies

$$\frac{d\left(\frac{f(v)}{1-F(v)}\right)}{dv} \geq 0.$$

Originally, the conditional probability density $f(t)/[1-F(t)]$ is known as the *Hazard Rate* or *Failure Rate* that approximates the probability that some component that smoothly works until time t breaks down by the "next" point in time (in $[t, t+dt]$). Within this context, the assumption of a declining failure rate over time seems quite intuitive. Although its extension to the auction context is not straightforward in terms of interpretation, the decisive advantage of the property is to provide additional structure of the distribution of valuations which allows an elegant identification of optimal auction designs. If the cumulative distribution function is strictly increasing, a nondecreasing density function is sufficient for the monotone hazard

[9]The given definition stems from Fudenberg and Tirole (1991, p. 267).

property due to the strictly decreasing denominator. Thus, the uniform distribution, widely used in auction theoretical and experimental applications, exhibits this property.[10]

3.1 Characterization of optimal auctions with public outside options

Theorem **??** characterizes optimal auction designs through the optimal selection of the marginal bidder. It can be interpreted as the optimal choice of the allocation probability. In symmetric equilibrium, the probability of allocation is equal to the probability that at least one realized valuation among those of N buyers is not smaller than the marginal bidder, i.e. $1 - F(\underset{\sim}{v})^N$. Due to the fact that the cumulative distribution function is strictly increasing, $f(v) > 0$, any feasible marginal bidder $\underset{\sim}{v} \in [v_-, v_+]$ precisely corresponds to one allocation probability and vice versa. If a seller selects a design that implies marginal bidder $\underset{\sim}{v} = v_+$, then the probability of allocation equals zero since not even a buyer with the largest valuation prefers to participate in the auction. In contrast, a design implying $\underset{\sim}{v} = v_-$ attracts any valuation type and the allocation probability is equal to one.

Theorem **??** indicates that more valuable public outside options have the same effect on the optimal marginal bidder as a variation of the seller's reservation value v_o by the same amount. If there is no possibility that the net valuation of any buyer exceeds the seller's reservation value, $v_o \geq v_+ - w$, she always sets a prohibitively high marginal bidder $\underset{\sim}{v}^* = v_+$ that is essentially equivalent to not offering the auction since, in this case, the allocation probability equals zero. Intuitively, a larger outside option value implies a larger cost for the seller since she has to compensate the auction winner for giving up his outside option. In order to avoid that larger cost, the seller intends to decrease the probability that she has to pay the compensation to an auction winner and, as a consequence, she increases the marginal bidder decreasing the allocation probability.

Theorem 3 *For any regular distribution function exhibiting the monotone ratio property, any auction design in class $\mathcal{A}$ that implies the optimal marginal bidder $\underset{\sim}{v}^*$ maximizes the expected revenue in equilibrium where $\underset{\sim}{v}^*$ is uniquely determined by:*

$$v_o + w + \frac{1 - F(\underset{\sim}{v}^*)}{f(\underset{\sim}{v}^*)} - \underset{\sim}{v}^* = 0, \quad \text{if } v_- - 1/f(v_-) < v_o + w < v_+, \tag{11}$$

where $\underset{\sim}{v}^* \in (v_-, v_+)$ in the preceding case, or

$$\underset{\sim}{v}^* = \begin{cases} v_-, & \text{if } v_o + w \leq v_- - 1/f(v_-) \\ v_+, & \text{if } v_o + w \geq v_+ \end{cases} .$$

[10]The exponential distribution exhibits the monotone hazard rate property, too, although its density function is strictly decreasing in valuations.

Proof. If the seller offers an auction, then she receives the expected equilibrium revenue $R^*(v_\sim)$. The probability that no buyer has a valuation exceeding that of the marginal bidder is $F(v_\sim)^N$. Then, the auctioned object is not allocated to a buyer but to the seller receiving her reservation value v_o. Thus, the seller's expected utility in equilibrium is given by $U_o(\cdot)$:

$$U_o(v_\sim) = R^*(v_\sim) + v_o\, F(v_\sim)^N.$$

Substitution of $R^*(v_\sim)$ by (**??**) and subsequent differentiation leads to

$$U_o'(v_\sim) = N\, F(v_\sim)^{N-1} f(v_\sim) \underbrace{\left[v_o + w + \frac{1 - F(v_\sim)}{f(v_\sim)} - v_\sim\right]}_{=:g(v_\sim)}.$$

Obviously, the distribution's properties imply for the defined function $g(v_\sim)$

$$g(v_-) = v_o + w + 1/f(v_-) - v_- \quad \text{and}$$
$$g(v_+) = v_o + w - v_+$$

and the Monotone Hazard Rate Property implies [11]

$$g'(v_\sim) < 0.$$

Since any $v_\sim > v_-$ implies $F(v_\sim)^{N-1} > 0$, it follows that the coefficient of $g(v_\sim)$ in $U'(v_\sim)$ is strictly positive due to $N > 0$ and $f(\cdot) > 0$, thus, the sign of $g(v_\sim)$ determines the sign of derivative $U_o'(v_\sim)$ for $v_\sim > v_-$. The derivative $U_o'(\cdot)$ vanishes at $v_\sim = v_-$. Depending on the particular parameter configuration, there are three different cases: (I) The configuration $v_o + w \leq v_- - 1/f(v_-)$ implies $g(v_-) \leq 0$. Since $g(\cdot)$ is maximized at $v_\sim = v_-$ due to $g' < 0$, $g(\cdot)$ is always strictly negative for any $v_\sim > v_-$. It follows that $U_o'(v_\sim) < 0$ for any $v_\sim > v_-$ and, thus, $v_\sim^* = v_-$ maximizes the seller's revenue. (II) The configuration $v_o + w \geq v_+$ implies $g(v_+) \geq 0$. Since $g(\cdot)$ is minimized at $v_\sim = v_+$ due to $g' < 0$, it follows that $g(\cdot)$ is strictly positive for any $v_\sim < v_+$ and nonnegative at $v_\sim = v_+$. Thus, $U_o'(v_\sim) > 0$ for any $v_\sim \in (v_-, v_+)$ and $v_\sim^* = v_+$ maximizes seller's payoff. (III) Any other configuration not coverd by cases I and II implies $g(v_-) > 0 > g(v_+)$. From the intermediate value theorem follows the existence of a unique solution of $g(v_\sim^*) = 0$ with $v_\sim^* \in (v_-, v_+)$ due to $g' < 0$ as implicitly defined by expression (**??**). The solution $v_\sim^*$ is a global maximizer since $U_o'(v_\sim) > 0$ if $v_\sim \in (v_-, v_\sim^*)$ and $U_o'(v_\sim) < 0$ if $v_\sim \in (v_\sim^*, v_+)$ follows from the behavior of $g(\cdot)$; notice that there is another stationary point of the objective function $U_o(\cdot)$ at v_-, since its derivative vanishes there independently of $g(\cdot)$. However, this must be a minimum for the considered case III and also for case II due to $U_o'(v_\sim) > 0$ for $v_\sim \in (v_-, v_\sim^*)$.

□

[11] The term $[1 - F(v_\sim)]\,/f(v_\sim)$ is the inverse of the Hazard Rate and thus decreasing. Since v_o and w are arbitrary constants and the argument negatively enters the sum, $g(v_\sim)$ is strictly monotonic decreasing.

Example 4 (UD) *Valuations of buyers are uniformly distributed over* $[v_-, v_+]$*:* $F(v_i) = \frac{v_i - v_-}{v_+ - v_-}$*. According to theorem* **??***:*

$$v_{\sim}^* = \frac{v_o + w + v_+}{2}, \quad \text{if } \frac{v_o + w + v_+}{2} \in (v_-, v_+)$$

or

$$v_{\sim}^* = \begin{cases} v_-, & \text{if } v_o + w + v_+ \leq 2v_- \\ v_+, & \text{if } v_o + w + v_+ \geq 2v_+ \end{cases}.$$

3.2 Optimal auctions and naive competition

The intensity of competition is identified by the value of the public outside option. To see the reasonability of this link, consider the seller's perspective. From his point of view, alternative opportunities to receive some substitute for her object offered in the auction represent competing offers due to buyers' single-object demand. Clearly, execution of any of these alternative transaction opportunities creates some return to the executor. Here, this value of execution is given by w and is the same for all buyers. It can be thought of as some equilibrium payoff generated by some other (not modeled) auction or bargaining game, too. Competition is naive since the value from executing some alternative transaction opportunity is exogenous and, in particular, does not interact with the behavior of the auction seller.

Uneducated common sense suggests that a revenue-maximizing seller might respond to fiercer competition with a more competitive auction. In the wake of competition, she could set a lower minimum bid or require a smaller entry cost than in the absence of competition such that a larger pool of buyers is attracted to the auction. However, rigorous analysis shows that the optimizing seller responds with a less competitive auction in the sense that she chooses a smaller allocation probability than without outside options. Proposition **??** provides the formal result.

Proposition 1 *For an interior solution determined by* $g(v_{\sim}^*) = 0$*, the marginal bidder increases with the value of the public outside option unless the probability of allocation is zero, i.e.* $v_{\sim}^* = v_+$*:*

$$\frac{dv_{\sim}^*}{dw} = \begin{cases} \left. \frac{1}{1 - d\left(\frac{1-F(v_{\sim})}{f(v_{\sim})}\right)/dv_{\sim}} \right|_{v_{\sim} = v_{\sim}^*} > 0, & \text{if } v_- - 1/f(v_-) \leq v_o + w < v_+ \\ 0, & \text{otherwise} \end{cases}.$$

Proof. Application of the implicit function theorem to (**??**) leads to the given derivative. Its sign is an implication of the monotone hazard rate property.

□

Example 5 (UD) *Valuations of buyers are uniformly distributed over* $[v_-, v_+]$: $F(v_i) = \frac{v_i - v_-}{v_+ - v_-}$. *Proposition* **??** *simplifies to*

$$\frac{dv^*_{\sim}}{dw} = \frac{1}{2}.$$

Corollary 2 *Any restriction of the pool of valuation types that participates in the auction reduces the probability of allocation.*

Proof. The (external) allocation probability[12] gives the probability that at least one realization among N drawn valuations exceeds that of the marginal bidder such that at least one bidder participates in the auction. Since $F(\cdot)$ is strictly monotonic increasing, the allocation probability $1 - F(v_{\sim})^N$ is strictly monotonic decreasing with the marginal bidder. Any additional restriction of the valuation pool that participates in the auction corresponds to a larger marginal bidder $v_{\sim}$.

□

4 Implementation of the optimal allocation probability

In the first-price and second-price auction, the marginal bidder is (in equilibrium) given by[13]

$$v_{\sim} = \max\{b_{\mathrm{M}} + w, v_-\}$$

Thus, the implementation of the optimal marginal bidder and the optimal allocation probability by a minimum bid is straightforward, in particular:

$$b_{\mathrm{M}} \begin{cases} \leq v_- - w, & \text{if } v_{\sim} = v_- \\ = v_{\sim} - w, & \text{if } v_{\sim} \in (v_-, v_+) \\ \geq v_+ - w, & \text{if } v_{\sim} = v_+ \end{cases} . \tag{12}$$

In the absence of outside options, the minimum bid precisely equals the marginal bidder. Consideration of outside option reduces the minimum bid by the value of the outside option. If a revenue-maximizing seller implements some participating valuation pool while ignoring the existence of valuable outside options, then she sets a minimum bid that is too large in the sense that it restricts a larger valuation pool from participating in the auction than she had intended to.

Interestingly, any increase of the public outside option value in the first-price or second-price auction leads to a lower competitiveness of the auction although a revenue-maximizing seller decreases the minimum bid. To see this, suppose $v^*_{\sim} \in (v_-, v_+)$. The derivative db_{M}/dw

[12] The residual probability gives the probability that the object remains with the seller.
[13] See Reiß (2005).

is comprises the optimal response of the seller to the increase in value w and the adjusted behavior of buyers. If the outside option value increases, a larger valuation pool prefers to not participate in the auction. This is reflected by an increasing marginal bidder. If the seller intends to implement the original marginal bidder, then she has to decrease the minimum bid by the same amount by which the value of the outside option has grown, see (**??**). However, this total decrease is suboptimal for the seller since

$$\left.\frac{db_{\mathrm{M}}}{dw}\right|_{v_{\text{-}}=v_{\text{-}}^{*}} = \frac{dv_{\text{-}}^{*}}{dw} - 1$$

and equivalently (proposition **??**)

$$\left.\frac{db_{\mathrm{M}}}{dw}\right|_{v_{\text{-}}=v_{\text{-}}^{*}} = \left.\frac{1}{1 - d\left(\frac{1-F(v_{\text{-}})}{f(v_{\text{-}})}\right)/dv_{\text{-}}}\right|_{v_{\text{-}}=v_{\text{-}}^{*}} - 1 \in (-1, 0].$$

Due to the monotone hazard rate property, the numerator can not be smaller than one. It follows that the optimal minimum bid decreases as the outside option value increases. However, the optimal minimum bid does not fall sufficiently to restore the valuation pool that originally preferred to participate in the auction (this requires $db_{\mathrm{M}}^{*}/dw = -1$) and, thus, the allocation probability and the competitiveness of the auction decreases. The following example illustrates.

Example 6 *Valuations of buyers are uniformly distributed over* $[v_-, v_+]$: $F(v_i) = \frac{v_i - v_-}{v_+ - v_-}$. *From above,* $dv_{\text{-}}^{*}/dw = 0.5$ *and* $db_{\mathrm{M}}^{*}/dw = 0.5 - 1 = -0.5$. *Now, the same finding is illustrated by comparison of two scenarios that differ in the value of outside options:*
(I) Canonical case: $w = 0$ *and* $V_i \sim U[0,1]$, $v_o = 0$. *Thus* $v_{\text{-}}^{*} = 0.5$. *The marginal bidder is implemented with minimum bid* $b_M = 0.5$. *With two buyers, the probability of allocation is* 0.75.
(II) Competition: $w = 0.2$ *and* $V_i \sim U[0,1]$, $v_o = 0$. *Then,* $v_{\text{-}}^{*} = 0.6$. *The marginal bidder is implemented with the minimum bid* $b_M = 0.4$. *Here, the allocation probability with two buyers is* 0.64.

5 Allocative Efficiency

In the canonical model, there is no outside option, $w = 0$, and usually $v_o = v_-$.[14] Thus, in the canonical parameter framework, any potential buyer has a valuation which exceeds that of the seller. It follows that every allocation that does not lead to a sale is Pareto-inefficient. According to theorem **??**, the canonical parameter framework implies that the optimal marginal bidder $v_{\text{-}}^{*}$ strictly exceeds the smallest feasible valuation v_- implying that the probability that the object is not allocated to some buyer is strictly positive. Thus, optimal auction designs in

[14] Cases with $v_o < v_-$ can be ruled out by an opportunity cost argument: the seller anticipates that she could simply set the posted price v_- and receive this revenue with certainty.

the canonical case always lead to inefficient allocations with positive probability. In contrast, consideration of outside options, and, also, separation of assumptions about the reservation value of the seller from assumptions about the valuation pool of buyers, demonstrates that optimal auction designs do not necessarily imply Pareto-inefficient allocations. There is always allocative efficiency, even with optimal auctions, if the inequality $v_o + w \leq v_- - 1/f(v_-)$ is satisfied and an optimizing seller does not restrict the valuation pool that participates in the auction, i.e. $v_-^* = v_-$. Less valuable outside options and smaller reservation values of the seller favor nonviolation of the inequality. In the extreme, the seller might have a cost of disposal for the object that she seeks to auction off leading to a negative reservation value. On the other hand, buyer-sided extremes are plausible, too. Potential buyers might be badly in need of either the auctioned object or some alternative object which may be a rather unsatisfying alternative such that the "natural" lower boundary $w \geq 0$ for outside option values may not need to hold and negative outside options result.[15] Optimal auctions are trivially efficient if the seller's reservation value exceeds the largest net valuation $v_o > v_+ - w$ such that the seller sets the allocation probability equal to zero. This is efficient since the largest valuation type never pays more than the amount $v_+ - w$ for the auctioned object while the the seller has a larger reservation value.

[15]By definition, any buyer values his best outside option at w. For most applications, it appears reasonable to assume $w \geq 0$ since the status quo of the buyer remains intact if he neither wins the auction nor executes the outside options. However, sometimes the conservation of the status quo leads to an extremely negative outcome where the assumption of some "best" negative outside option that slightly improves on the status quo is justified. These scenarios are equivalent to punishments of buyers that are unsuccessful in the auction.

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